Categories for Quantum Computing

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Category Theory?

- Category theory is an abstract field of mathematics. Best explained through analogies.
  - Many logicians like to prove things about proving things.
  - Many category theorists do mathematics about doing mathematics.
- It can be used to talk about structural features of mathematical theories.
- A lot of mathematics for quantum theory, e.g. linear-algebra, functional analysis, operator algebras, noncommutative geometry.
- One can also use category theory to talk about structural or compositional features of quantum computation or quantum theory.
- **Remark:** I do not aim to do everything with category theory only. My aim is to use category theory to identify good compositional features relevant to quantum theory and quantum computation.



The Curry-Howard-Lambek Correspondence

The Curry-Howard-Lambek correspondence can be understood (and made precise) as a correspondence between

$$\text{Type Systems} \longleftrightarrow \text{Logic} \longleftrightarrow \text{Category Theory/Mathematics}$$

For example, we have the following correspondences:

- λ-calculi $\longleftrightarrow$ Intuitionistic Logics $\longleftrightarrow$ Cartesian Closed Categories
- Unit type $\longleftrightarrow$ Atomic Truth $\longleftrightarrow$ Terminal Object
- Empty type $\longleftrightarrow$ Atomic Falsity $\longleftrightarrow$ Initial Object
- Sum types $\longleftrightarrow$ Disjunction $\longleftrightarrow$ Coproducts
- Linear λ-calculi $\longleftrightarrow$ Intuitionistic Multiplicative Linear Logics $\longleftrightarrow$ Symmetric Monoidal Closed Categories
- Linear $\lambda\mu$-style-calculi $\longleftrightarrow$ Multiplicative Linear Logics $\longleftrightarrow$ $*$-autonomous Categories

Good Mathematical Models $\Longrightarrow$ Good Design?

Historical examples where good mathematical models has produced good type system/logic design:

- Monads in category theory $\Longrightarrow$ Monads in type systems/functional programming. Ideas described by Eugenio Moggi in 1980s–1990s.
- Coherence spaces $\Longrightarrow$ Linear Logic. Paper by Jean-Yves Girard in 1988.

Inspired by these developments and the Curry-Howard-Lambek correspondence, we ask the question:

What are good mathematical models for quantum computation from which we can extract good logic/type system design?

Design Principle

- There are two main picture of quantum theory/quantum computation – the Heisenberg picture and the Schrödinger picture.
- The two pictures are *dual* to each other.
- This duality can be made mathematically precise and presented at different levels of refinement. Relevant mathematics: linear-algebra, functional analysis, operator algebras, noncommutative geometry.
- Linear Logic and its categorical models are well suited for describing dualities.
- We consider mathematical models relevant to quantum theory where the LL duality coincides with the Heisenberg-Schrödinger duality on relevant objects.
- We want:
  - **Schrödinger picture** $\leftrightarrow$ **positive logical polarity**.
  - **Heisenberg picture** $\leftrightarrow$ **negative logical polarity**.

Good Duality $\Longrightarrow$ Good Higher-order Structure

- In Classical Logic, the implication ($\Longrightarrow$) can be decomposed via negation ($\neg$) and the conjuction ($\wedge$) or disjunction ($\vee$).

$$A \Longrightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$$

- In Classical Linear Logic, the linear implication ($\multimap$) can be decomposed via the linear negation $(-)^{\perp}$ and the multiplicative conjuction ($\otimes$) or multiplicative disjunction ($\parr$).

$$A \multimap B \equiv A^{\perp} \parr B \equiv (A \otimes B^{\perp})^{\perp}$$

- The word "classical" should not be understood as being in dichotomy with the word "quantum". It is in dichotomy with the word "intuitionistic".
- The difference between classical and intuitionistic logic lies in double negation. In classical logic $A \iff \neg\neg A$, whereas in intuitionistic logic we have $A \Longrightarrow \neg\neg A$.

Heisenberg-Schrödinger Duality Logically?

- In Classical Linear Logic $\vdash A \multimap B$ iff $\vdash B^{\perp} \multimap A^{\perp}$.
- If we have $\vdash A \multimap B$ and $\vdash B \multimap C$, then also $\vdash A \multimap C$.
- Dually, we then have $\vdash B^{\perp} \multimap A^{\perp}$ and $\vdash C^{\perp} \multimap B^{\perp}$ and also $\vdash C^{\perp} \multimap A^{\perp}$. Notice that to derive the third proof from the previous two requires composing backwards (compared to previous bullet point).

Heisenberg-Schrödinger Duality Logically?

- Schrödinger picture: forwards evolution of "state". e.g. $\mathcal{U}_S(\rho) = U\rho U^*$.
- Heisenberg picture: backwards modification of "observable", e.g. $\mathcal{U}_H(O) = U^* O U$.
- Regardless of choice of picture, the expectation value after unitary evolution (Heisenberg vs Schrödinger style) does not change: let ρ be a mixed state and let O be an observable (self-adjoint operator) both on the same Hilbert space, then $\mathrm{tr}(\mathcal{U}_S(\rho)O) = \mathrm{tr}(U\rho U^* O) = \mathrm{tr}(\rho U^* O U) = \mathrm{tr}(\rho \mathcal{U}_H(O))$.
- Perform unitary evolution U_1 followed by U_2 on state $\rho \mapsto U_2 U_1 \rho U_1^* U_2^*$ (Schrödinger) $\leftrightarrow$ modify observable O by apllying U_2 first then U_1, i.e. $O \mapsto U_1^* U_2^* O U_2 U_1$ (Heisenberg).
- Notice that the order of application in the Heisenberg picture is reversed compared to the Schrödinger one.

Baby's First Interpretation of the Heisenberg-Schrödinger Duality in Classical Linear Logic

Baby's First Interpretation of the Heisenberg-Schrödinger Duality in Classical Linear Logic

The following is not fully precise, but semi-OK for a baby intuition:

- Atomic Types A $\leftrightarrow$ Atomic Propositions A $\leftrightarrow$ spaces $\mathbb{M}_n$
- Dual Atomic Types $A^\perp \leftrightarrow$ Negated Atomic Propositions $A^\perp \leftrightarrow$ dual spaces $\mathbb{M}_n^*$
- Basic Derivations $A \vdash B \leftrightarrow$ Basic Proofs $A \vdash B \leftrightarrow$ CPTP maps $\mathbb{M}_n \to \mathbb{M}_m$ (Schrödinger superoperators)
- Dual Derivations $B^\perp \vdash A^\perp \leftrightarrow$ Proof of negation $B^\perp \vdash A^\perp \leftrightarrow$ CPU maps $\mathbb{M}_m \to \mathbb{M}_n$ (Heisenberg superoperators)
- For such kinds of derivations/proofs, LL duality $\equiv$ Heisenberg-Schrödinger duality.
- **Remark:** This can be refined and made more precise via polarised linear logic.

Plan

This mini course will proceed as follows:

- Introduce a lot of mathematical background relevant to quantum computation and quantum theory.
- Introduce relevant categorical concepts and how they appear in categories relevant to quantum computation.
- We wish to use mathematical and categorical structures which behave well w.r.t. Heisenberg-Schrödinger duality.
- We aim to capture both pictures (Schrödinger and Heisenberg) in such a way that the logical/categorical duality coincides with the Heisenberg-Schrödinger duality on appropriate objects.
- As we progress, the mathematical ideas become more refined and complicated. Not enough time to provide full details for the more advanced concepts, but I hope to provide some level of intuition and motivation.

Normed Spaces

Definition

A *normed space* is a complex vector space X equipped with a norm $\|-\|: X \to [0, \infty)$, i.e. a function which satisfies the following properties:

- for every $x \in X$ and $\lambda \in \mathbb{C}$, we have $\|\lambda x\| = |\lambda| \|x\|$.
- for every $x, y \in X$, we have $\|x + y\| \leq \|x\| + \|y\|$.
- for every $x \in X$, we have $\|x\| = 0$ iff $x = 0$.

Some quick facts:

- The canonical metric (or distance) is defined by $d(x, y) = \|y - x\|$.
- The norm (or metric) can be used to define a topology, called the *norm topology*.
- For a finite-dimensional vector space X, every norm on X induces the same topology.

Examples of Normed Spaces

Example

- The zero dimensional vector space 0 equipped with its unique norm.
- The complex numbers $\mathbb{C}$ with norm $\|\lambda\| \triangleq |\lambda|$.
- Consider the vector space $\mathbb{C} \oplus \mathbb{C}$. Consider the following three norms:
  - $\|(\alpha, \beta)\| = |\alpha| + |\beta|$. This is the ℓ^1-norm. We write $\mathbb{C} \overset{1}{\oplus} \mathbb{C}$ for this space.
  - $\|(\alpha, \beta)\| = \max(|\alpha|, |\beta|)$. This is the ℓ^∞-norm. We write $\mathbb{C} \overset{\infty}{\oplus} \mathbb{C}$ for this space.
  - $\|(\alpha, \beta)\| = \sqrt{|\alpha|^2 + |\beta|^2}$. This is the ℓ^2-norm. We write $\mathbb{C}^2$ for this (Hilbert) space.
- We write $\mathbb{C}^n$ for the standard Hilbert space $\mathbb{C}^n$. Its norm is the usual ℓ^2-norm, i.e.

$$\|(\lambda_1, \ldots, \lambda_n)\| \triangleq \sqrt{|\lambda_1|^2 + \cdots + |\lambda_n|^2}$$

Examples of Normed Spaces (Mixed-State Quantum Computation)

Example

- Consider the vector space $\mathbb{M}_n$, consisting of the $n \times n$ complex matrices. The *operator norm* on $\mathbb{M}_n$ is defined by

$$\|A\| \triangleq \sup\{\|Av\| : v \in \mathbb{C}^n, \|v\| \leq 1\}.$$

  We write M_n for this normed space.

- Consider the same vector space $\mathbb{M}_n$. A matrix $A \in \mathbb{M}_n$ is *positive* (also known as *positive semi-definite*) iff there exists a matrix $B \in \mathbb{M}_n$, such that $A = B^*B$. For every matrix $A \in \mathbb{M}_n$, there exists a unique positive matrix $|A|$ with the property that $|A|^2 = A^*A$. The *trace norm* on $\mathbb{M}_n$ is defined by

$$\|A\| \triangleq \mathrm{tr}(|A|)$$

  We write T_n for the this normed space. If P is positive, then $\|P\| = \mathrm{tr}(P)$.

Banach Spaces

Definition

A *Banach space* is a normed space X which is complete with respect to the topology induced by the norm. More specifically, this means that every Cauchy sequence[1] $(x_n)_{n\in\mathbb{N}}$ in X has a topological limit, i.e. there exists $x \in X$, such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

Example

- Every finite-dimensional normed space.
- Every Hilbert space H with its canonical ℓ^2-norm $\|h\| \triangleq \sqrt{\langle h, h\rangle}$.

[1] $\forall \epsilon > 0. \exists N \in \mathbb{N}. \forall i, j > N. \|x_i - x_j\| < \epsilon$

More Examples of Banach Spaces

Example

- The vector space

$$\ell^1 \triangleq \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mid \sum_{i \in \mathbb{N}} |\lambda_i| < \infty \right\}$$

  is an (infinite-dimensional) Banach space when equipped with the ℓ^1-norm $\|(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)\| \triangleq \sum_{i \in \mathbb{N}} |\lambda_i|$.

- The vector space

$$\ell^\infty \triangleq \{(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mid \sup\{|\lambda_i| : i \in \mathbb{N}\} < \infty\}$$

  is an (infinite-dimensional) Banach space when equipped with the ℓ^∞-norm $\|(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)\| \triangleq \sup_i |\lambda_i|$.

Hilbert spaces and quantum theory

- Hilbert spaces are fundamental for quantum theory and quantum computation.
- Pure state quantum computation can be formulated using Hilbert spaces.
- Pure quantum states $\Longleftrightarrow$ normalised vectors in Hilbert spaces.
- Quantum operations, in pure state quantum computation, include:
  - Applying unitary maps to quantum states.
  - Preparation of auxiliary states (aka ancillas).
- Quantum operations do not increase the norm of a vector.

Inner-product spaces

Definition

An inner-product space is a complex vector space V equipped with an *inner-product*, i.e. a function $\langle -, = \rangle \colon V \times V \to \mathbb{C}$, such that:

- $\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle$.
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
- $\langle v, v \rangle \geq 0$.
- If $\langle v, v \rangle = 0$, then $v = 0$.

Example

The canonical inner-product on $\mathbb{C}^n$ is defined by

$$\langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle \triangleq \sum_i v_i \overline{w_i}$$

Hilbert Spaces

Definition

A *Hilbert space* H is a complex inner-product space H, such that H is a Banach space w.r.t. the ℓ^2-norm $\|h\| \triangleq \sqrt{\langle h|h\rangle}$.

Example

Every finite-dimensional inner-product space is a Hilbert space.

Proposition

Every finite-dimensional Hilbert space is isomorphic to $\mathbb{C}^n$ *for some* $n \in \mathbb{N}$.

Infinite-dimensional Hilbert Spaces

Example

The vector space

$$\ell^2 \triangleq \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mid \sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty \right\}$$

is a Hilbert space when equipped with the inner-product

$$\langle (\lambda_1, \ldots, \lambda_n, \ldots), (\mu_1, \ldots, \mu_n, \ldots) \rangle \triangleq \sum_{i \in \mathbb{N}} \lambda_i \overline{\mu_i}.$$

Infinite-dimensional Hilbert Spaces

Example

- For any set X, the space

$$\ell^2(X) \triangleq \left\{ f : X \to \mathbb{C} \mid \sum_{x \in X} |f(x)|^2 < \infty \right\}$$

  with inner product $\langle f, g \rangle \triangleq \sum_{x \in X} f(x)\overline{g(x)}$ is a Hilbert space.
- If we define $|x\rangle\,(x) = 1$ and $|x\rangle\,(y) = 0$ for $x \neq y$, then the set $\{|x\rangle\}_{x \in X}$ is an orthonormal basis of $\ell^2(X)$.
- If X has cardinality n, then $\ell^2(X) \cong \mathbb{C}^n$. If X has cardinality ω, then $\ell^2(X) \cong \ell^2$.

Proposition

Every Hilbert space is isomorphic to $\ell^2(X)$ for some set X.

Bounded Operators

Definition

Let X and Y be Banach spaces and let $f : X \to Y$ be a linear map (aka operator). We say that f is:

- *bounded* iff there exists $r \geq 0$ such that $\|f(x)\| \leq r\|x\|$ for all $x \in X$;
- *continuous* iff f is continuous w.r.t the norm topologies of X and Y;
- a *contraction* iff $\|f(x)\| \leq \|x\|$ for all $x \in X$;
- an *isometry* iff $\|f(x)\| = \|x\|$ for all $x \in X$.

A few facts:

- injection $\Leftarrow$ isometry $\Rightarrow$ contraction $\Rightarrow$ bounded $\iff$ continuous;
- if X finite-dimensional, then bounded $\iff$ linear $\iff$ continuous.

Examples of Bounded Operators

Example

- Every linear map $f\colon X \to Y$ between finite-dimensional Banach spaces.
- Recall the Banach space $\ell^1 \triangleq \left\{(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mid \sum_{i\in\mathbb{N}} |\lambda_i| < \infty\right\}$ with the ℓ^1-norm $\|(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)\| \triangleq \sum_{i\in\mathbb{N}} |\lambda_i|$. We can define a bounded operator

$$\mathsf{tr}\colon \ell^1 \to \mathbb{C} :: (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mapsto \sum_{i\in\mathbb{N}} \lambda_i.$$

In fact, tr is a contraction, because

$$\|\mathsf{tr}(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)\| = \left|\sum_{i\in\mathbb{N}} \lambda_i\right| \leq \sum_{i\in\mathbb{N}} |\lambda_i| = \|(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)\|$$

Non-example of a Bounded Operator

- Recall the Banach space $\ell^\infty \triangleq \{(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mid \sup\{|\lambda_i| : i \in \mathbb{N}\} < \infty\}$ with the ℓ^∞-norm $\|(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)\| \triangleq \sup_i |\lambda_i|$. The assignment

$$\mathsf{tr} \colon \ell^\infty \to \mathbb{C} :: (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \mapsto \sum_{i \in \mathbb{N}} \lambda_i$$

  is not even a function, let alone a bounded one. Indeed,

$$\mathsf{tr}(1, 1, \ldots, 1, \ldots) = \infty$$

  so tr is not a function with the indicated type.

Adjoints and Unitaries for Hilbert Spaces

Assume that H_1 and H_2 are Hilbert spaces. Then:

- If $f\colon H_1 \to H_2$ is bounded, then there exists a unique map $f^*\colon H_2 \to H_1$ called the *adjoint* of f such that $\langle h_2, f(h_1)\rangle = \langle f^*(h_2), h_1\rangle$ for all $h_1 \in H_1$ and $h_2 \in H_2$.
- A *unitary* operator is a bounded operator $f\colon H_1 \to H_2$ such that $f^* \circ f = \mathrm{id}_{H_1}$ and $f \circ f^* = \mathrm{id}_{H_2}$.
- Every unitary operator is an isometry.
- All Hilbert spaces of the same dimension are unitarily isomorphic to each other.
- **Example:** $\ell^2(\{t, f\}) \cong \mathbb{C}^2$ and $\ell^2(\mathbb{N}) \cong \ell^2(\mathbb{Z}) \cong \ell^2(\mathbb{Q})$.
- **Warning:** An orthonormal basis is not always a vector space basis (e.g. in infinite-dimensions)!

Adjoints and Unitaries for $\mathbb{C}^n$

Assume that $H_1 = \mathbb{C}^n$ and $H_2 = \mathbb{C}^m$ are Hilbert spaces. A linear map $f : \mathbb{C}^n \to \mathbb{C}^m$ is uniquely determined by an $m \times n$ matrix. So we can identify f with its matrix representation, say A. For such Hilbert spaces:

- The adjoint f^* is given by the conjugate transpose A^*.
- Unitary operators $\iff$ unitary matrices, i.e. matrices A such that $AA^* = A^*A = I_n$.

Examples of Unitaries

Some unitary operators that we can define on $\mathbb{C}^2$:

- $H|0\rangle \triangleq \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle; \quad H|1\rangle \triangleq \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$
- $X|0\rangle \triangleq |1\rangle \quad X|1\rangle \triangleq |0\rangle.$

Examples of Unitaries (infinite-dimensions)

Some unitary operators that we can define on $\ell^2(\mathbb{N})$:

- $H|0\rangle \triangleq \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$; $\quad H|1\rangle \triangleq \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$; $\quad H|n\rangle = |n\rangle$ otherwise.
- $K|0\rangle \triangleq e^{\frac{i\pi}{4}}|0\rangle$; $\quad K|n\rangle = |n\rangle$ otherwise.

Some unitary operators that we can define on $\ell^2(\mathbb{Z})$:

- $S|n\rangle \triangleq |n+1\rangle$.
- $P|n\rangle \triangleq |n-1\rangle$. Obviously, $S^* = P$.

A unitary $N\colon \ell^2(\mathbb{R}) \to \ell^2(\mathbb{I})$ by $N|r\rangle \triangleq |f(r)\rangle$ where $f : \mathbb{R} \to \mathbb{I}$ is any bijection...

Positivity and Complete Positivity (Concretely)

- An element (matrix) $p \in \mathbb{M}_n$ is called *positive* if $\langle h, ph \rangle \geq 0$ for every $h \in \mathbb{C}^n$.
- Equivalently, $p \in \mathbb{M}_n$ is *positive* iff $p = a^* a$ for some $a \in \mathbb{M}_n$.
- We say that a linear function $\varphi \colon \mathbb{M}_n \to \mathbb{M}_m$ is *positive* if it preserves positive elements, i.e. if $p \in \mathbb{M}_n$ is positive, then $\varphi(p) \in \mathbb{M}_m$ is also positive.
- We say that a linear function $\varphi \colon \mathbb{M}_n \to \mathbb{M}_m$ is *completely positive* if the function

$$\mathbb{M}_{kn} \cong \mathbb{M}_k \otimes \mathbb{M}_n \xrightarrow{\mathrm{id} \otimes \varphi} \mathbb{M}_k \otimes \mathbb{M}_m \cong \mathbb{M}_{km}$$

  is positive for every $k \in \mathbb{N}$.
- We say that a linear function $\varphi \colon \mathbb{M}_n \to \mathbb{M}_m$ is *unital* if $\varphi(I_n) = I_m$, i.e. if it preserves the unit.
- We say that a linear function $\varphi \colon \mathbb{M}_n \to \mathbb{M}_m$ is *trace-preserving* if $\mathrm{tr}(\varphi(a)) = \mathrm{tr}(a)$, i.e. if it preserves the trace.

Positivity and Complete Positivity (Heisenberg Picture)

- Observe that $\mathbb{M}_n \cong B(\mathbb{C}^n) = \{f : \mathbb{C}^n \to \mathbb{C}^n \mid f \text{ is bounded}\}$.
- More generally, let H be a Hilbert space and consider $B(H) = \{f : H \to H \mid f \text{ is bounded}\}$. This is a Banach space w.r.t operator norm.
- An element (operator) $p \in B(H)$ is called *positive* if $\langle h, ph\rangle \geq 0$ for every $h \in H$.
- Equivalently, $p \in B(H)$ is *positive* iff $p = a^*a$ for some $a \in B(H)$.
- A linear map $\varphi : B(H_1) \to B(H_2)$ is *positive* if it preserves positive elements.
- Complete positivity may be defined in a similar way (details omitted).
- A linear map $\varphi : B(H_1) \to B(H_2)$ is *unital* if $\varphi(\mathrm{id}_{H_1}) = \mathrm{id}_{H_2}$, i.e. if it preserves the identity operator.
- The spaces $B(H)$ and M_n (both equipped with operator norm) are relevant for the Heisenberg picture.

Positivity and Complete Positivity (Schrödinger Picture)

- For $a \in B(H)$, one can define $\mathrm{tr}(a) = \sum_i \langle ae_i, e_i \rangle$ for any ONB of H. This is independent of the choice of ONB.
- For $a \in B(H)$, one can again define $|a| \in B(H)$ to be the unique positive operator such that $|a|^2 = a^*a$.
- The *trace class* is the subset $T(H) = \{t \in B(H) : \mathrm{tr}(|t|) < \infty\} \subseteq B(H)$. This is a Banach space w.r.t trace norm $\|t\| \triangleq \mathrm{tr}(|t|)$ and inherited vector space structure.
- Positive elements in $T(H)$ are defined in the same way as in $B(H)$.
- A linear map $\varphi\colon T(H_1) \to T(H_2)$ is *positive* if it preserves positive elements.
- Complete positivity may be defined in a similar way (details omitted).
- A linear map $\varphi\colon T(H_1) \to T(H_2)$ is *trace-preserving* if $\mathrm{tr}(\varphi(a)) = \mathrm{tr}(a)$, for every $a \in T(H)$, i.e. if it preserves the trace.
- The spaces $T(H)$ and T_n (both equipped with trace norm) are relevant for the Schrödinger picture.

Positivity and Complete Positivity (Abstractly)

- More generally, (complete) positivity can be defined for C*-algebras (and von Neumann algebras). This works well for the Heisenberg picture.
- Even more generally, (complete) positivity can be defined for certain kinds of matrix ordered spaces. This definition works for both pictures.

Quantum Operations in the Schrödinger Picture

- Let $f : \mathbb{M}_n \to \mathbb{M}_m$ be a linear map.
- We say that f is a *quantum operation (aka quantum channel)* in the Schrödinger picture if f is a completely positive and trace-preserving (CPTP) map.
- In the Schrödinger picture, a matrix $\rho \in \mathbb{M}_n$ is called a *density matrix* if $\mathrm{tr}(\rho) = 1$ and $0 \leq \rho$, i.e. ρ is positive.
- Equivalently, density matrices may be identified as the (C)PTP maps $\varphi : \mathbb{C} \to \mathbb{M}_n$.
- **Example:** $\frac{1}{3} |0\rangle \langle 0| + \frac{2}{3} |1\rangle \langle 1|$ is a density matrix in $\mathbb{M}_2$.
- **Example:** If U is a $n \times n$ unitary matrix, then $U(-)U^* : \mathbb{M}_n \to \mathbb{M}_n$ describes unitary evolution in the Schrödinger picture on $\mathbb{M}_n$.
- **Example:** Quantum measurement in the computational basis in $\mathbb{M}_2$ can be described by the function $\mathbb{M}_2 \to \mathbb{M}_2 :: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$
- All of these functions are CPTP maps.

Quantum Operations in the Heisenberg Picture

- Let $f : \mathbb{M}_n \to \mathbb{M}_m$ be a linear map.
- We say that f is a *quantum operation (aka quantum channel)* in the Heisenberg picture if f is a completely positive and unital (CPU) map.
- A *state*, in the Heisenberg picture, is a CPU map $\varphi : \mathbb{M}_n \to \mathbb{C}$.
- Every such state is of the form $\varphi = \mathrm{tr}(\rho -)$, with $\rho \in \mathbb{M}_n$ a density matrix.
- **Example:** If U is a $n \times n$ unitary matrix, then $U^*(-)U : \mathbb{M}_n \to \mathbb{M}_n$ describes unitary evolution in the Heisenberg picture on $\mathbb{M}_n$.
- **Example:** Quantum measurement in the computational basis in $\mathbb{M}_2$ can be described by the function $\mathbb{M}_2 \to \mathbb{M}_2 :: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$
- All of these functions are CPU maps.
- **Note:** Composition in the Heisenberg picture is reversed (i.e. backwards).

Category Theory

Definition

A locally small category $\mathbf{C}$ consists of the following data:

- A collection (e.g. a set) of objects for which we write $\mathrm{Ob}(\mathbf{C})$;
- For every pair of objects A, B, a set of morphisms $\mathbf{C}(A, B)$. We write $f : A \to B$ to indicate that $f \in \mathbf{C}(A, B)$;
- For every object A, a morphism $\mathrm{id}_A \in \mathbf{C}(A, A)$, called *identity*;
- For every pair of morphisms $g \in \mathbf{C}(B, C)$ and $f \in \mathbf{C}(A, B)$, a morphism $(g \circ f) \in \mathbf{C}(A, C)$. We say that $g \circ f$ is the *composition* of g and f;

such that

- Composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$;
- Composition is unital, i.e. for $f : A \to B$, we have that $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

The Category **Set**

Example

The category **Set** has:

- sets as objects, i.e. $\mathrm{Ob}(\mathbf{Set})$ is the collection of all sets (which is not a set).
- functions as morphisms, i.e. a morphism $f : X \to Y$ in **Set** is simply a function from X to Y.
- the identity function as the identity morphism, i.e. $\mathrm{id}_X : X \to X :: x \mapsto x$.
- composition of functions as the composition operation, i.e. $g \circ f$ is simply function composition.

Example

Addition $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a morphism in **Set**. The partial function $f(x) \triangleq \frac{1}{x} : \mathbb{R} \to \mathbb{R}$ is not a morphism in **Set**. The function $g(x) \triangleq \frac{1}{x} : (\mathbb{R} - \{0\}) \to \mathbb{R}$ is a morphism in **Set**.

The Category **FHilb**

Example

The category **FHilb** has:

- finite-dimensional Hilbert spaces as objects;
- linear functions as morphisms;
- the identity function as the identity morphism;
- composition of functions as the composition operation;

Example

The Hadamard operation $H\colon \mathbb{C}^2 \to \mathbb{C}^2 :: \begin{cases} |0\rangle \mapsto |+\rangle \\ |1\rangle \mapsto |-\rangle \end{cases}$ is a morphism in **FHilb**.

Every unitary map $u\colon H \to K$ is a morphism in **FHilb**. Furthermore, if $\lambda \in \mathbb{C}$, then $\lambda u\colon H \to K$ is also a morphism in **FHilb**, even though this linear function can be norm-increasing.

The Category **FHilb**

Example

The category **FHilb** has:

- finite-dimensional Hilbert spaces as objects;
- linear functions as morphisms;
- the identity function as the identity morphism;
- composition of functions as the composition operation;

Example

- Let $v \in \mathbb{C}^n$. Then the linear map $1 \mapsto v \colon \mathbb{C} \to \mathbb{C}^n$ is a morphism in **FHilb**. Note that v does not have to be normalised.
- Non-trivial quantum measurements are not morphisms in **FHilb**.

The Category $\mathbf{FHilb_c}$

Example

The category $\mathbf{FHilb_c}$ has:

- finite-dimensional Hilbert spaces as objects;
- linear contractions as morphisms;
- the identity function as the identity morphism;
- composition of functions as the composition operation;

Example

- Let $v \in \mathbb{C}^n$. Then the linear map $1 \mapsto v \colon \mathbb{C} \to \mathbb{C}^n$ is a morphism in $\mathbf{FHilb_c}$ iff $\|v\| \leq 1$.
- Unitaries and isometries (between f.d. Hilbert spaces) are morphisms in $\mathbf{FHilb_c}$.
- Quantum operations cannot increase the norm, so this category has fewer irrelevant morphisms.

Categories for pure state quantum computation

Example

- **Hilb** – Hilbert spaces and bounded maps between them.
- $\mathbf{Hilb_c}$ – Hilbert spaces and linear contractions between them.
- $(\mathbf{F})\mathbf{Hilb_i}$ – (finite-dimensional) Hilbert spaces and linear isometries between them.
- $(\mathbf{F})\mathbf{Hilb_u}$ – (finite-dimensional) Hilbert spaces and unitaries between them.
- These categories are relevant in the context of pure-state quantum computation.
- **Remark:** Both **Hilb** and **FHilb** contain a lot of irrelevant morphisms for quantum computation.

Categories for mixed state quantum computation

Example

The category **CPM** has:

- matrix spaces $\mathbb{M}_n$ as objects;
- the morphisms $\varphi\colon \mathbb{M}_n \to \mathbb{M}_m$ are the completely positive maps;
- the identity function as the identity morphism;
- composition of functions as the composition operation;

Example

- All CPTP and CPU maps $\mathbb{M}_n \to \mathbb{M}_m$ are morphisms in **CPM**.
- Therefore both Heisenberg and Schrödinger quantum operations are morphisms.

Categories mixed state quantum computation

Example

The category **CPM** has:

- matrix spaces $\mathbb{M}_n$ as objects;
- the morphisms $\varphi \colon \mathbb{M}_n \to \mathbb{M}_m$ are the completely positive maps;
- the identity function as the identity morphism;
- composition of functions as the composition operation;

Problem: CPM also contains non-physical morphisms, e.g. morphisms that can be norm-increasing:

- The CP map $2 \cdot \mathrm{id} \colon \mathbb{M}_2 \to \mathbb{M}_2$ increases the norm of any non-zero vector, but it is a valid morphism in **CPM**.

Categories for mixed state quantum computation

Example

The category **S** has:

- matrix spaces $\mathbb{M}_n$ as objects;
- the morphisms $\varphi\colon \mathbb{M}_n \to \mathbb{M}_m$ are the CPTP maps;
- the identity functions as the identity morphisms;
- composition of functions as the composition operation;
- Therefore the Schrödinger quantum operations are morphisms, hence the name.

Categories for mixed state quantum computation

Example

The category **H** has:

- matrix spaces $\mathbb{M}_n$ as objects;
- the morphisms $\varphi\colon \mathbb{M}_n \to \mathbb{M}_m$ are the CPU maps;
- the identity functions as the identity morphisms;
- composition of functions as the composition operation;
- Therefore the Heisenberg quantum operations are morphisms, hence the name.

Problem: Composition in the Heisenberg picture is supposed to be backwards.
Solution: Take the *opposite* category.

Opposite Category

Definition

Let $\mathbf{C}$ be a category. The *opposite* category of $\mathbf{C}$, for which we write $\mathbf{C}^{\mathrm{op}}$, is given by the following data:

- The objects are the same as those of $\mathbf{C}$.
- A morphism $f\colon A \to B$ in $\mathbf{C}^{\mathrm{op}}$ is a morphism $f\colon B \to A$ in $\mathbf{C}$, i.e. $\mathbf{C}^{\mathrm{op}}(A, B) = \mathbf{C}(B, A)$.
- Identities are the same as in $\mathbf{C}$.
- Composition of $f \in \mathbf{C}^{\mathrm{op}}(A, B)$ and $g \in \mathbf{C}^{\mathrm{op}}(B, C)$ is given by $(f \circ g) \in \mathbf{C}^{\mathrm{op}}(A, C)$, i.e. $g \circ_{\mathbf{C}^{\mathrm{op}}} f = f \circ_{\mathbf{C}} g$.

Categories for mixed state quantum computation

Example

The category $\mathbf{H}^{\mathrm{op}}$ has:

- matrix spaces $\mathbb{M}_n$ as objects;
- the morphisms $\varphi\colon \mathbb{M}_n \to \mathbb{M}_m$ are the CPU maps $\mathbb{M}_m \to \mathbb{M}_n$;
- the identity functions as the identity morphisms;
- backward composition of functions as the composition operation;
- This resolves the aforementioned problem.

Baby's First Formulation of the Heisenberg-Schrödinger Duality

A (very) simple form of the Heisenberg-Schrödinger duality can be stated as follows:

There is a bijection between CPTP maps $\mathbb{M}_n \to \mathbb{M}_m$ and CPU maps $\mathbb{M}_m \to \mathbb{M}_n$.

Remark: If we represent the (linear) CPTP/CPU maps as matrices (w.r.t. the usual basis of $\mathbb{M}_n$), then the bijection is given by linear-algebraic transposition.

Remark: Moreover, this bijection is determined via the following universal property: for a linear map $\varphi\colon \mathbb{M}_n \to \mathbb{M}_m$, there exists a unique linear function $\varphi^t\colon \mathbb{M}_m \to \mathbb{M}_n$, such that for every $a \in \mathbb{M}_n$, $b \in \mathbb{M}_m$,

$$\operatorname{tr}(\varphi(a)b) = \operatorname{tr}\big(a\varphi^t(b)\big)$$

Then $(-)^t$ is the aforementioned bijection.

The Heisenberg-Schrödinger Duality Categorically?

We can formulate this duality using categorical terminology. Towards this end, we introduce some new concepts.

Functors in Category Theory

Definition

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F\colon \mathbf{C} \to \mathbf{D}$ is given by the following data:

- A map on objects $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$. We write $F(A)$ or FA for the action of F on an object $A \in \mathrm{Ob}(\mathbf{C})$.
- For each pair of objects $A, B \in \mathrm{Ob}(\mathbf{C})$, a map $\mathbf{C}(A,B) \to \mathbf{D}(FA, FB)$. We write $F(f)$ or Ff for the action of F on a morphism $f \in \mathbf{C}(A,B)$.

Moreover, this data has to satisfy the following properties:

- F preserves identities: $F(\mathrm{id}_A) = \mathrm{id}_{FA}$.
- F preserves composition: $F(g \circ f) = Fg \circ Ff$.

Functors in Category Theory

Example

- On any category $\mathbf{C}$, the identity functor $\mathrm{Id}\colon \mathbf{C} \to \mathbf{C}$. It is defined by $\mathrm{Id}(A) \triangleq A$ and $\mathrm{Id}(f) \triangleq f$.
- The powerset functor $P\colon \mathbf{Set} \to \mathbf{Set}$. It is defined on objects by

$$P(X) \triangleq \{Y \mid Y \subseteq X\}$$

and for a morphism $f\colon X \to Y$, it is defined by

$$P(f)\colon PX \to PY :: S \mapsto f[S].$$

Functors in Category Theory

Example

- The forgetful functor $U\colon \mathbf{FHilb} \to \mathbf{Set}$. On objects, we define $U(H) \triangleq H$, i.e. we take the underlying set of H. For a morphism $f\colon X \to Y$, we define $U(f) \triangleq f$, i.e. we take the same function.
- The unit ball functor $\mathrm{Ball}\colon \mathbf{Hilb_c} \to \mathbf{Set}$. On objects, we define $\mathrm{Ball}(H) \triangleq \{h \in H : \|h\| \leq 1\}$. For a linear contraction $f\colon H_1 \to H_2$, we define $\mathrm{Ball}(f) \triangleq f|_{\mathrm{Ball}(H_1)}$, i.e. we take the (co)restriction of f to the unit balls.

Functors in Category Theory

Example

- Fix $n \in \mathbb{N}$. Then $(\mathbb{C}^n \otimes -)\colon \mathbf{FHilb} \to \mathbf{FHilb}$ is a functor. We define it on objects by the assignment $H \mapsto \mathbf{C}^n \otimes H$ and on a morphism $f\colon H \to K$, we define it by the assignment $f \mapsto (\mathrm{id} \otimes f)\colon (\mathbb{C}^n \otimes H) \to (\mathbb{C}^n \otimes K)$.

A non-example of a functor is given by the following. Consider the unit ball assignment $\mathrm{Ball}\colon \mathbf{FHilb} \to \mathbf{Set}$. On objects, we define $\mathrm{Ball}(H) \triangleq \{h \in H : \|h\| \leq 1\}$. However, for a linear function (not necessarily a contraction) $f\colon H_1 \to H_2$, we cannot define $\mathrm{Ball}(f) \triangleq f|_{\mathrm{Ball}(H_1)}$, because f can potentially increase the norm. In other words, f does not necessarily admit a (co)restriction to the unit balls of H_1 and H_2.

Isomorphisms in Category Theory

Definition

Let $\mathbf{C}$ be a category and $f\colon A \to B$ be a morphism. We say that f is an *isomorphism*, if there exists a (necessarily unique) morphism $f^{-1}\colon B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

Example

We can classify the isomorphisms in some of the aforementioned categories:

- In **Set**, the categorical isomorphisms coincide with the bijections.
- In **Hilb** and **FHilb**, the categorical isomorphisms coincide with the (bounded) linear isomorphisms.
- In $\mathbf{Hilb_c}$ and $\mathbf{FHilb_c}$, the categorical isomorphisms coincide with the linear isometric isomorphisms which in turn coincide with the unitary maps.

Isomorphism of Categories

Definition

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. We say that F is an *isomorphism (of categories)*, if there exists a (necessarily unique) functor $F^{-1}\colon \mathbf{D} \to \mathbf{C}$, such that $F^{-1} \circ F = \mathrm{Id}_A$ and $F \circ F^{-1} = \mathrm{Id}_B$. We say that two categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic*, and we write $\mathbf{C} \cong \mathbf{D}$ to indicate this, if there exists an isomorphism of categories $F\colon \mathbf{C} \to \mathbf{D}$.

Example

By taking adjoints, we get the following isomorphisms:

- $\mathbf{Hilb} \cong \mathbf{Hilb}^{\mathrm{op}}$.
- $\mathbf{FHilb} \cong \mathbf{FHilb}^{\mathrm{op}}$.
- $\mathbf{Hilb_c} \cong \mathbf{Hilb_c}^{\mathrm{op}}$.
- $\mathbf{FHilb_c} \cong \mathbf{FHilb_c}^{\mathrm{op}}$.

Isomorphism of Categories

Definition

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. We say that F is an *isomorphism (of categories)*, if there exists a (necessarily unique) functor $F^{-1}\colon \mathbf{D} \to \mathbf{C}$, such that $F^{-1} \circ F = \mathrm{Id}_A$ and $F \circ F^{-1} = \mathrm{Id}_B$. We say that two categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic*, and we write $\mathbf{C} \cong \mathbf{D}$ to indicate this, if there exists an isomorphism of categories $F\colon \mathbf{C} \to \mathbf{D}$.

Remark: One can consider the category **Cat** which has small categories as objects and functors between them as morphisms. Then the isomorphisms in **Cat** coincide with the isomorphism of small categories from the above definition.

Remark: The word *small* above can be made precise and it is used to avoid Russel style paradoxes.

The Heisenberg-Schrödinger Duality Categorically

We can now formulate a simple form of the Heisenberg-Schrödinger Duality in categorical terms. Recall that **S** has matrix spaces $\mathbb{M}_n$ as objects and CPTP maps as morphisms, whereas **H** has the same objects, but CPU maps as morphisms.

Theorem

There exists an isomorphism of categories $F\colon \mathbf{S} \cong \mathbf{H}^{\mathrm{op}}$ *defined by*

$$F(\mathbb{M}_n) \triangleq \mathbb{M}_n$$

$$F(f) \triangleq f^t.$$

In other words, the isomorphism is given by transposition.

Monoidal and Tensor Products

- **Motivation:** In order to obtain stronger compositional properties, it is important to consider tensor products of quantum systems.
- Let's begin with simple cases relevant to finite-dimensional quantum theory.

Tensor Products for Hilbert Spaces

Definition

Let H and K be f.d. Hilbert spaces. Their tensor product, $H \overset{2}{\otimes} K$, is the Hilbert space given by the following data:

- Its vector space structure coincides with the vector space tensor product $H \otimes K$.
- The inner product is determined by the assignment

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle \triangleq \langle h_1, h_2 \rangle \langle k_1, k_2 \rangle$$

  and extending (anti)linearly in the relevant component.

Remark: Tensor products of infinite-dimensional Hilbert spaces can also be defined. This requires taking the completion of the vector space tensor product $H \otimes K$ with respect to the induced ℓ^2-norm of the inner-product. This justifies our notation.

Tensor Products for Hilbert Spaces

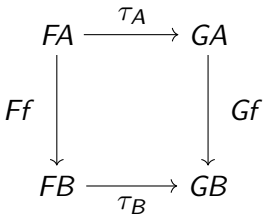
- In fact, $(- \overset{2}{\otimes} =) \colon \mathbf{FHilb} \times \mathbf{FHilb} \to \mathbf{FHilb}$ is a functor when we define $f \overset{2}{\otimes} g \triangleq f \otimes g$, i.e. the functorial action coincides with the vector space tensor.
- The same assignment also gives us a functor $(- \overset{2}{\otimes} =) \colon \mathbf{FHilb_c} \times \mathbf{FHilb_c} \to \mathbf{FHilb_c}$.
- This tensor product further satisfies (in both categories):
  - $H \overset{2}{\otimes} K \cong K \overset{2}{\otimes} H$.
  - $H \overset{2}{\otimes} \mathbb{C} \cong H$.
  - $\mathbb{C} \overset{2}{\otimes} H \cong H$.
  - $(H \overset{2}{\otimes} K) \overset{2}{\otimes} L \cong H \overset{2}{\otimes} (K \overset{2}{\otimes} L)$.
- Similar properties are enjoyed by other tensor products in other categories.
- (Symmetric) Monoidal Categories allow us to capture many of these shared properties.

Natural Transformations

Before we may introduce symmetric monoidal categories, we need an additional categorical notion.

Definition

Let $F, G \colon \mathbf{C} \to \mathbf{D}$ be two functors. A *natural transformation* $\tau \colon F \Rightarrow G$ is a family of morphisms $\{\tau_A \colon FA \to GA \mid A \in \mathrm{Ob}(\mathbf{C})\}$, indexed by the objects of $\mathbf{C}$, such that for every morphism $f \colon A \to B$ in $\mathbf{C}$, the following diagram



commutes, i.e. $\tau_B \circ Ff = Gf \circ \tau_A$. If each component τ_A is an isomorphism, then we say that τ is a *natural isomorphism*.

Examples of Natural Transformations

Example

Consider the category **FHilb**.

- The symmetry $\sigma_{H,K} \colon H \overset{2}{\otimes} K \to K \overset{2}{\otimes} H :: h \otimes k \mapsto k \otimes h$ is a natural isomorphism.
- The left unitor $\lambda_H \colon \mathbb{C} \overset{2}{\otimes} H \to H :: a \otimes h \mapsto ah$ is a natural isomorphism.
- The right unitor $\rho_H \colon H \overset{2}{\otimes} \mathbb{C} \to H :: h \otimes a \mapsto ah$ is a natural isomorphism.
- The associator $\alpha_{H,K,L} \colon H \overset{2}{\otimes} (K \overset{2}{\otimes} L) \to (H \overset{2}{\otimes} K) \overset{2}{\otimes} L :: h \otimes (k \otimes l) \mapsto (h \otimes k) \otimes l$ is a natural isomorphism.

If we replace **FHilb** with $\mathbf{FHilb_c}$, then the same holds.

Symmetric Monoidal Categories

Definition

A *(symmetric) monoidal category* is given by the following data:

- A category $\mathbf{C}$.
- A functor $(-\otimes =)\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$, called *monoidal product*.
- An object $I\in \mathrm{Ob}(\mathbf{C})$, called *monoidal unit*.
- A natural isomorphism $\lambda_A\colon I\otimes A\xrightarrow{\cong} A$, called *left unitor*.
- A natural isomorphism $\rho_A\colon A\otimes I\xrightarrow{\cong} A$, called *right unitor*.
- A natural isomorphism $\alpha_{A,B,C}\colon A\otimes(B\otimes C)\xrightarrow{\cong}(A\otimes B)\otimes C$, called *associator*.
- (A natural isomorphism $\sigma_{A,B}\colon A\otimes B\xrightarrow{\cong} B\otimes A$, called *symmetry*.)

This data is required to satisfy some coherence conditions expressed in terms of commutative diagrams, but we omit this for brevity.

Symmetric Monoidal Categories

Example

Some examples of symmetric monoidal categories:

- **FHilb** with the data from previous slides.
- $\mathbf{FHilb_c}$ with the data from previous slides.
- **CPM** with monoidal unit $I \triangleq \mathbb{C}$ and monoidal product defined by $\mathbb{M}_n \otimes \mathbb{M}_m \triangleq \mathbb{M}_{nm}$. The associator, left unitor, and right unitor now coincide with the identity function. The symmetry is given by

$$\sigma_{\mathbb{M}_n,\mathbb{M}_m} : \mathbb{M}_{nm} \to \mathbb{M}_{mn} :: \sum_{i,j} c_{ij}(A_i \otimes B_j) \mapsto \sum_{i,j} c_{ij}(B_j \otimes A_i), \text{ where } A_i \in \mathbb{M}_n, B_j \in \mathbb{M}_m$$

- The categories $\mathbf{S}, \mathbf{H}$, and $\mathbf{H}^{\mathrm{op}}$, are also symmetric monoidal. The symmetric monoidal structure is defined similarly to that of **CPM**.

Symmetric Monoidal Categories

Example

The category **Set** is also symmetric monoidal.

- The monoidal product is given by the cartesian product $X \otimes Y \triangleq X \times Y$.
- The monoidal unit is given by a choice of a singleton set $I \triangleq 1 = \{1\}$.
- The left unitor is the bijection $\lambda_X : 1 \times X \xrightarrow{\cong} X :: (1, x) \mapsto x$.
- The right unitor is the bijection $\rho_X : X \times 1 \xrightarrow{\cong} X :: (x, 1) \mapsto x$.
- The associator is the bijection
  $\alpha_{X,Y,Z} : X \times (Y \times Z) \xrightarrow{\cong} (X \times Y) \times Z :: (x, (y, z)) \mapsto ((x, y), z)$.
- The symmetry is the bijection $\sigma_{X,Y} : X \times Y \xrightarrow{\cong} Y \times X :: (x, y) \mapsto (y, x)$.

In fact, **Set** is also a *cartesian* category, which is a simple special case of a symmetric monoidal category.

Higher-order Computation

- The simply-typed lambda calculus is a model of higher-order computation.
- For (higher-order) quantum computation, it is more appropriate to use a *linear* lambda calculus.
- From the Curry-Howard-Lambek isomorphism, we already know that

  $$\text{Simply-typed Lambda Calculi} \longleftrightarrow \text{Cartesian Closed Categories.}$$

- **Question:** What does the Curry-Howard-Lambek isomorphism give us for linear lambda calculi?
- **Answer:**

  $$\text{Simply-typed Linear Lambda Calculi} \longleftrightarrow \text{Symmetric Monoidal Closed Categories.}$$

Currying in **Set**

Consider the category **Set**.

- For sets X and Y, we write $[X \to Y] \triangleq \{f : X \to Y \mid f \text{ is a function}\}$ for the set of functions from X to Y.
- We have a bijection, called *currying*,

$$\Lambda : \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, [Y \to Z]) :: f \mapsto (x \mapsto (f(x, -))).$$

- In other words, if $f : X \times Y \to Z$ is a function on two arguments, then $\Lambda(f) : X \to [Y \to Z]$ is a function that takes an element of X and returns a function from Y to Z.
- Obvious link to currying in functional programming and higher-order computation.
- The category **Set** is cartesian closed (and therefore also symmetric monoidal closed).

Currying in **Vect**

Let **Vect** be the category of vector spaces (over $\mathbb{C}$) with linear maps as morphisms.

- **Vect** is symmetric monoidal when equipped with the usual tensor product $\otimes$ and $\mathbb{C}$ acting as the monoidal unit. The associator, symmetry and unitors are defined in the obvious way.
- Let $L(V, W)$ be the set of linear maps between vector spaces V and W. It inherits a vector space structure from W in a pointwise way, e.g. $(\lambda_1 f + \lambda_2 g)(v) \triangleq \lambda_1 f(v) + \lambda_2 g(v)$.
- We consider $L(V, W)$ as a vector space, so it is an object of **Vect**.
- We now have a bijection, called currying, which is given by

$$\Lambda \colon \mathbf{Vect}(X \otimes Y, Z) \cong \mathbf{Vect}(X, L(Y, Z)) :: f \mapsto (x \mapsto (f(x, -))).$$

- The category **Vect** is monoidal closed.

Symmetric Monoidal Closed Categories

Definition

A *symmetric monoidal closed* category (SMCC) is a symmetric monoidal category $(\mathbf{C}, \otimes, I)$, such that for every object $B \in \mathrm{Ob}(\mathbf{C})$, there exists a functor $B \multimap - \colon \mathbf{C} \to \mathbf{C}$ with the property that there exists a bijection

$$\Lambda \colon \mathbf{C}(A \otimes B, C) \cong \mathbf{C}(A, B \multimap C)$$

which is natural in A and C (details omitted).

- **Intuition:** One may think of $B \multimap C$ as a function space object in $\mathbf{C}$.
- $B \multimap C$ is called an *internal hom*, whereas $\mathbf{C}(B, C)$ is called an *external hom*. The former is an object, whereas the latter is a set.
- If $\mathbf{C}$ is a SMCC, then there is a canonical way to extend $\multimap$ to a functor $(- \multimap =) \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ such that Λ becomes natural in all three components.

Symmetric Monoidal Closed Categories

Example

A few examples of symmetric monoidal closed categories:

- The category **Set**.
- The category **Vect**.
- The category **FVect**, of finite-dimensional vector spaces with linear maps as morphisms.
- The category **FHilb** (details later).
- The category **CPM** (details later).

Remark: I don't know of any proofs that the categories **Hilb**, $\mathbf{Hilb_c}$, $\mathbf{FHilb_c}$, **S**, **H**, and $\mathbf{H}^{\mathrm{op}}$ are SMCC. I am pretty confident that they are not SMCC, but I have not spent much time trying to verify this. In any case, we can see that monoidal *closure* is harder to obtain.

Symmetric Monoidal Closure of **FVect**

Example

Consider the category **FVect**, whose objects are the f.d. vector spaces (over $\mathbb{C}$) and whose morphisms are the linear maps.

- The symmetric monoidal closed structure is the same as for **Vect**.
- The internal-hom $L(V, W)$ admits another equivalent definition.
- If V is a f.d. vector space, then the dual $V^* \triangleq L(H, \mathbb{C})$ is also a vector space.
- There is a canonical linear isomorphism $\iota \colon V^* \otimes W \cong L(V, W) :: \varphi \otimes w \mapsto (v \mapsto \varphi(v)w)$.
- In other words, the internal-hom in **FVect** can be decomposed using $\otimes$ and $(-)^*$. The same is *not* true for **Vect**.

Symmetric Monoidal Closure of **FHilb**

Example

Consider the category **FHilb**.

- If H is a f.d. Hilbert space, then the dual space $H^* \triangleq L(H, \mathbb{C})$ can also be equipped with a Hilbert space structure by using the Riesz representation theorem. If $\varphi\colon H \to \mathbb{C}$ is a bounded linear functional, then $\varphi = \langle -, h_\varphi \rangle$, for a unique vector $h_\varphi \in H$. Then H^* is a Hilbert space with inner-product

$$\langle \varphi, \psi \rangle = \langle h_\psi, h_\varphi \rangle$$

- Then **FHilb** is a SMCC with internal-hom $H \multimap K \triangleq H^* \overset{2}{\otimes} K$. Currying is given by

$$\Lambda\colon \mathbf{FHilb}(H \overset{2}{\otimes} K, L) \cong \mathbf{FHilb}(H, K^* \overset{2}{\otimes} L) :: f \mapsto (h \mapsto \iota^{-1}(f(h, -)))$$

Symmetric Monoidal Closure of **CPM**

Example

Consider the category **CPM**.

- **Intuition:** For an object $\mathbb{M}_n$, we have an obvious choice of basis. This allows us to fix a linear isomorphism $\mathbb{M}_n \cong \mathbb{M}_n^*$, so we can identify $\mathbb{M}_n$ and its dual space $\mathbb{M}_n^*$.
- We define the internal-hom by $\mathbb{M}_n \multimap \mathbb{M}_m \triangleq \mathbb{M}_{nm}$. Intuition: $\mathbb{M}_n^* \otimes \mathbb{M}_m = \mathbb{M}_{mn}$.
- Currying can be defined in a similar way as for **FHilb**.

Curry-Howard-Lambek Again

- We saw that the simply-typed lambda calculus can be extended with sum types.
- This allows for conditional branching, e.g. `if` statements, pattern matching.
- Sum types can also be added to a linear lambda calculus.
- Under the Curry-Howard-Lambek correspondence, we have

$$\text{sum types} \leftrightarrow \text{coproducts}$$

Initial Objects

Definition

An *initial object* in a category **C** is an object $\varnothing$, such that for every object $A \in \mathrm{Ob}(\mathbf{C})$, there exists a unique morphism $\varnothing \to A$.

Fact: Any two initial objects are isomorphic to each other. If A is initial and $A \cong B$, then B is also initial.

Example

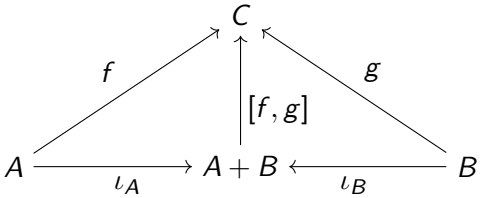
The following are examples of initial objects:

- In **Set**, the initial object is the empty set $\varnothing$.
- In (**F**)**Vect**, the zero-dimensional vector space 0 is an initial object.
- In (**F**)**Hilb**, the zero-dimensional Hilbert space 0 is an initial object.
- In **CPM**, the object $\mathbb{M}_0$ is an initial object.

Binary Coproducts

Definition

Let $\mathbf{C}$ be a category. A *binary coproduct* of two objects A and B is given by the following data: an object $A + B$, together with two morphisms $\iota_A \colon A \to A + B$ and $\iota_B \colon B \to A + B$ such that for any object C and morphisms $f \colon A \to C$ and $g \colon B \to C$, there exists a unique morphism $[f, g] \colon A + B \to C$ with the property that the following diagram



commutes.

Finite Coproducts

Definition

We say that a category $\mathbf{C}$ has *finite coproducts* if it has an initial object and binary coproducts.

Example

The following categories have finite coproducts:

- In $\mathbf{Set}$, binary coproducts are given by disjoint union $X + Y \triangleq X \times \{0\} \cup Y \times \{1\}$.
- In $(\mathbf{F})\mathbf{Vect}$, binary coproducts are given by direct sum $V \oplus W$.
- In $(\mathbf{F})\mathbf{Hilb}$, binary coproducts are given by direct sum $H \oplus K$.

The categories $\mathbf{CPM}, (\mathbf{F})\mathbf{Hilb_c}, \mathbf{S}, \mathbf{H}, \mathbf{H}^{\mathrm{op}}$ do not have binary coproducts.

Curry-Howard-Lambek Again

- In Linear Logic, one has the *multiplicative conjunction* $\otimes$, but also the *additive conjunction* $\&$.
- Under the Curry-Howard-Lambek correspondence, we know that we can model the additive conjunction via categorical products in certain kinds of categories.

Terminal Objects

Definition

A *terminal object* in a category **C** is an object 1, such that for every object $A \in \mathrm{Ob}(\mathbf{C})$, there exists a unique morphism $A \to 1$.

Fact: Any two terminal objects are isomorphic to each other. If A is terminal and $A \cong B$, then B is also terminal.

Example

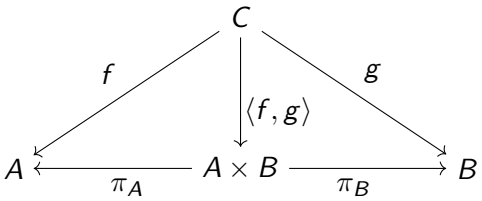
The following are examples of terminal objects:

- In **Set**, any singleton set is a terminal object.
- In (**F**)**Vect**, the zero-dimensional vector space 0 is a terminal object.
- In (**F**)**Hilb**, the zero-dimensional Hilbert space 0 is a terminal object.
- In **CPM**, the object $\mathbb{M}_0$ is a terminal object.

Binary Products

Definition

Let $\mathbf{C}$ be a category. A *binary product* of two objects A and B is given by the following data: an object $A \times B$, together with two morphisms $\pi_A \colon A \times B \to A$ and $\pi_B \colon A \times B \to B$ such that for any object C and morphisms $f \colon C \to A$ and $g \colon C \to B$, there exists a unique morphism $\langle f, g \rangle \colon C \to A \times B$ with the property that the following diagram



commutes.

Finite Products

Definition

We say that a category $\mathbf{C}$ has *finite products* if it has a terminal object and binary products.

Example

The following categories have finite products:

- In **Set**, binary products are given by cartesian product $X \times Y \triangleq \{(x, y) \mid x \in X, y \in Y\}$.
- In (**F**)**Vect**, binary products are given by direct sum $V \oplus W$.
- In (**F**)**Hilb**, binary products are given by direct sum $H \oplus K$.

The categories $\mathbf{CPM}, (\mathbf{F})\mathbf{Hilb}_{\mathbf{c}}, \mathbf{S}, \mathbf{H}, \mathbf{H}^{\mathrm{op}}$ do not have binary products.

What categorical problems do we run into?

With the fairly simplistic approach so far, a few problems arise:

- The suggested categories for mixed state quantum computation do not have (co)products.
- The suggested categories for infinite-dimensional pure state quantum computation are not closed.
- The categories **CPM** and **FHilb** contain too many non-physical morphisms.
- It appears there is no clean categorical way to represent classical information with the ideas so far.

Solution:

- Use more refined mathematics, e.g. C*-algebras, von Neumann algebras, functional analysis.
- Consider categories based on these ideas that avoid the aforementioned problems.

(Classical) Information for Mixed State Quantum Computation

- We can represent both classical and quantum information in mixed state quantum computation.
- Consider the set of 2×2 density matrices $P = \{\rho \in \mathbb{M}_2 \mid \mathrm{tr}(\rho) = 1 \text{ and } \rho \geq 0\}$.
- The subset $C = \left\{ \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \mid p_1 + p_2 = 1 \text{ and } p_1, p_2 \geq 0 \right\}$ can be seen as representing classical probability distributions of bits (booleans).
- Clearly, $C \subset P$ consists of matrices that are considerably simpler.
- The main idea (for f.d. quantum computation) is to consider not only spaces like $\mathbb{M}_n$, but also certain kinds of subspaces $V \subseteq \mathbb{M}_n$.
- How to do this in a clean way is already known: von Neumann algebras.
- This works well for the Heisenberg picture (even in infinite-dimensions) and it also works fairly well for f.d. quantum computation in both pictures.

Von Neumann Algebras

Definition

A f.d. *von Neumann algebra (vN-algebra)* is a f.d. complex vector space A together with:

- a bilinear binary operation, called *multiplication*, and written via juxtaposition, which is associative, i.e. $a(bc) = (ab)c$;
- an element $1 \in A$, called *unit*, such that $1a = a = a1$;
- an anti-linear unary operation $(\cdot)^*$, called *involution*, such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$;
- a (necessarily complete) norm which is *submultiplicative*, i.e. $\|ab\| \leq \|a\|\|b\|$, and which satisfies the *C*-identity*, i.e. $\|a^*a\| = \|a\|^2$.

Remark: Infinite-dimensional vN-algebras can also be defined, but we omit this here.

Von Neumann Algebras

Example

The vector space $\mathbb{M}_n$ is a vN-algebra with:

- Unit 1 given by the identity matrix.
- Multiplication (ab) is given by matrix multiplication.
- Involution $(-)^*$ is given by taking the conjugate transpose.
- The norm $\|-\|$ is the operator norm. Recall that we write M_n for this Banach space.

Von Neumann Algebras

Example

The vector space

$$\bigoplus_{1 \leq i \leq n} \mathbb{M}_{k_i}$$

is also a vN-algebra with unit, multiplication, and involution defined pointwise, and with norm given by the $\ell^\infty-$norm, i.e. $\|(a_1, \ldots, a_n)\| = \max(\|a_1\|, \ldots, \|a_n\|)$.
In fact, modulo a vN-algebra isomorphism, every f.d. vN-algebra is of this form.

Example

As a special case, if each $\mathbb{M}_{k_i} = \mathbb{C}$, then we see that $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ is also a vN-algebra. These kinds of f.d. vN-algebras allow us to represent classical information. Observe that multiplication for this vN-algebra is commutative, i.e. $ab = ba$.

Von Neumann Algebras and Classical Information

- A vN-algebra A is *commutative* iff $ab = ba$, for every $a, b \in A$.
- A f.d. vN-algebra A is *commutative* iff $A = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.
- Commutative vN-algebras $\longleftrightarrow$ classical information.
- Therefore, with vN-algebras, we can represent both classical and quantum information (and mixtures of the two).
- **Example:** The vN-algebra $\mathbb{C} \oplus \mathbb{C}$ represents the type of bits (or booleans) in mixed state quantum computation.
- **Example:** The vN-algebra $\mathbb{M}_2$ represents the type of qubits in mixed state quantum computation.
- Observe that one can embed $\mathbb{C} \oplus \mathbb{C}$ onto the diagonal of $\mathbb{M}_2$.

(Complete) Positivity for vN-algebras

- An element $p \in A$ of a vN-algebra is called *positive* if $p = a^*a$, for some $a \in A$.
- If A is a vN-algebra, then the vector space $\mathbb{M}_n(A)$ of matrices with elements in A is also a vN-algebra, with operations defined in the obvious way.
- If $f : A \to B$ is a linear map, then $f_n : \mathbb{M}_n(A) \to \mathbb{M}_n(B) :: [a_{ij}] \mapsto [f(a_{ij})]$ is again a linear function.
- We say that a linear map $f : A \to B$ is *completely positive* if f_n preserves positive elements for every $n \in \mathbb{N}$.
- **Remark:** There is a linear isomorphism $\mathbb{M}_n \otimes A \cong \mathbb{M}_n(A)$, and up to natural isomorphism, one may identify the map $f_n : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ with the map $\mathrm{id}_{\mathbb{M}_n} \otimes f : \mathbb{M}_n \otimes A \to \mathbb{M}_n \otimes B$.
- This definition of complete positivity is equivalent to the one we saw previously for maps $f : \mathbb{M}_n \to \mathbb{M}_m$. It is, of course, more general, because it covers more spaces.

Quantum Computation, VN-Algebras, and the Heisenberg Picture

VN-algebras can be used to describe a more refined notion of (f.d.) quantum computation in the Heisenberg picture.

- Let $f : A \to B$ be a linear map between two vN-algebras. We say that f is *unital* if $f(1_A) = 1_B$.
- We say that f is a *quantum operation (aka quantum channel)* in the Heisenberg picture if f is a completely positive and unital (CPU) map.
- **Example:** the classical "not" function can be described by the CPU map

$$\varphi : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \oplus \mathbb{C} :: (a, b) \mapsto (b, a)$$

- VN-algebras can be used to describe infinite-dimensional quantum computation as well (details omitted).

Nicer Categorical Properties?

- **Problem:** The category **H** does not have binary products.
- **Solution:** Use the category **vNAlg**, whose objects are the f.d. vN-algebras and whose morphisms are the CPU maps.
- The category **vNAlg** is also symmetric monoidal and it has finite products.
- The monoidal product of A and B is given by the vN-algebra $A \mathbin{\bar{\otimes}} B$ with:
  - underlying vector space $A \otimes B$.
  - unit $1_A \otimes 1_B$.
  - multiplication determined by $(a_1 \otimes b_1)(a_2 \otimes b_2) \triangleq (a_1 a_2) \otimes (b_1 b_2)$.
  - involution determined by $(a \otimes b)^* \triangleq a^* \otimes b^*$.
  - there is now a unique way to define a vN-algebra norm (details omitted).
- The binary product of A and B is the vN-algebra $A \oplus B$ with unit, multiplication, and involution defined componentwise, and equipped with the ℓ^∞-norm, i.e. $\|(a, b)\| \triangleq \max(\|a\|, \|b\|)$.

Heisenberg Picture Revisited

- **Problem:** But composition in the Heisenberg picture should be backwards.
- **Solution:** The category $\mathbf{vNAlg}^{\mathrm{op}}$ is symmetric monoidal and it has finite coproducts.
- Therefore, $\mathbf{vNAlg}^{\mathrm{op}}$ can be seen as a model of first-order mixed state quantum computation in the Heisenberg picture for finite dimensions.

Remark: There is a stronger notion of morphism between vN-algebras that preserves all of their structure. But this is less relevant for quantum computation.

What about the Schrödinger Picture?

- The Heisenberg-Schrödinger duality admits a refinement on the level of vN-algebras and their preduals.
- This requires a good deal of mathematics, even in the f.d. case, so I will take some shortcuts.
- The main ideas are the following:
  - vN-algebras and their morphisms describe the Heisenberg picture.
  - Every vN-algebra and every quantum operation in the Heisenberg picture is *dual* to a certain kind of space and certain kind of map, respectively.
  - There is a unique *predual* of both the vN-algebra and the quantum operation, respectively.
  - These preduals give the objects and the quantum operations in the Schrödinger picture.
  - For f.d. quantum computation, these preduals actually coincide with the duals.

Von Neumann Coalgebras

- Let A be a f.d. vN-algebra.
- Its dual space A^* can be equipped with a coalgebra structure:
  - Instead of a unit, we have a counit $\epsilon\colon A^* \to \mathbb{C}$.
  - Instead of a multiplication, we have a comultiplication $\delta\colon A^* \to A^* \overset{\text{h}}{\otimes} A^*$, where $\overset{\text{h}}{\otimes}$ is a special kind of monoidal product.
  - We again have a similar involution.
  - There is additional data involving norms.
  - This data satisfies conditions that are dual to that of a vN-algebra.
- The above definition can be made without references to vN-algebras.
- We call these kinds of objects f.d. *vN-coalgebras*.

Examples of Von Neumann Coalgebras

Recall that we write T_n for the Banach space $\mathbb{M}_n$ equipped with the trace norm[2].

Example

The space T_n is a vN-coalgebra with:

- counit given by the trace functional $\varepsilon \triangleq \mathrm{tr} : T_n \to \mathbb{C} :: t \mapsto \mathrm{tr}(t)$.
- involution given by the conjugate transpose
- comultiplication given by $\delta : T_n \to T_n \overset{\mathrm{h}}{\otimes} T_n :: e_{ij} \mapsto \sum_k (e_{kj} \otimes e_{ik})$, where $e_{ij} \in T_n$ is the matrix that has 1 in position (i, j) and 0 elsewhere.

[2] We actually need some more information to make this precise.

Examples of Von Neumann Coalgebras

Recall that we write T_n for the Banach space $\mathbb{M}_n$ equipped with the trace norm[3].

Example

We also have the following:

- If C and D are f.d. vN-coalgebras, then their direct sum $C \oplus D$ can be equipped with a vN-coalgebra structure. Its norm is now the ℓ^1-norm $\|(c,d)\| \triangleq \|c\| + \|d\|$.
- We have a duality $T_n \cong M_n^*$ and $T_n^* \cong M_n$.
- Recall that every f.d. vN-algebra is of the form $\bigoplus_{1 \leq i \leq n} M_{k_i}$.
- Every f.d. vN-coalgebra is of the form $\bigoplus_{1 \leq i \leq n} T_{k_i}$.
- So the duality exchanges: $M_n \longleftrightarrow T_n$, ℓ^∞-norms $\longleftrightarrow$ ℓ^1-norms, unit $\longleftrightarrow$ counit, multiplication $\longleftrightarrow$ comultiplication, etc.

[3] We actually need some more information to make this precise.

Morphisms between Von Neumann Coalgebras

Let C and D be f.d. vN-coalgebras.

- One can dualise the definition of a positive element in vN-algebra A to obtain the definition of a positive linear functional $\varphi\colon C \to \mathbb{C}$.
- Dualising further, one can define completely positive maps $f\colon C \to D$.
- The counit $\epsilon\colon C \to \mathbb{C}$, can be thought of as a trace. In case $C = T_n$, then it is precisely the trace. We say that a linear map $f\colon C \to D$ is *trace-preserving* if $\epsilon_D \circ f = \epsilon_C$.
- We can now talk about CPTP maps between f.d. vN-coalgebras and formulate the Schrödinger picture for f.d. quantum computation.

Heisenberg-Schrödinger Categorically

- We write **vNCoalg** for the category whose objects are f.d. vN-coalgebras and whose morphisms are CPTP maps between them.
- One can now prove that **vNCoalg** is symmetric monoidal and it has finite coproducts.
- The Heisenberg-Schrödinger duality can now be stated in a categorical form as the fact that there is a categorical duality $(-)^* : \mathbf{vNCoalg} \simeq \mathbf{vNAlg}^{\mathrm{op}} : (-)^*$.
- Therefore, both **vNCoalg** and **vNAlg** give us a model for first-order finite-dimensional quantum programming.

What about higher-order?

- Recall that **CPM** is symmetric monoidal closed.
- To get (co)products, we can consider the following category.
- Let $\mathbf{vNAlg_{cp}}$ be the category whose objects are f.d. vN-algebras and whose morphisms are completely positive maps.
- One can show that $\mathbf{vNAlg_{cp}}$ is symmetric monoidal closed and it has finite (co)products.
- **Remark:** $\mathbf{vNAlg_{cp}} \simeq \mathbf{CP}^*[\mathbf{CPM}]$, where $\mathbf{CP}^*[-]$ is a categorical construction.
- **Problem:** $\mathbf{vNAlg_{cp}}$, just like **CPM**, has morphisms that are not physical, e.g. $2 \cdot \mathrm{id}$.
- Main observation behind this problem: morphisms can increase the norm.
- **Solution:** consider categories where the morphisms cannot increase the norm, i.e. where the morphisms are contractive.

Analytic Properties of Quantum Systems

- Usual (commutative) analysis techniques are not ideal for quantum computation in a categorical sense.
- **Example:** The transposition function $(-)^T : M_n \to M_n$ is not only a contraction, it is even an isometric isomorphism. But it is not physical.
- Recall that $(-)^T : M_n \to M_n$ is positive, but it is not *completely* positive.
- It turns out that $(-)^T : M_n \to M_n$ is contractive, but not *completely* contractive.
- To define precisely what "complete contraction" means, we need even more mathematics...

Operator Spaces and Complete Contractions

- An *operator space* is a vector space X equipped with a family of norms $\|-\|_n : \mathbb{M}_n(X) \to \mathbb{R}$, one for each $n \in \mathbb{N}$, which satisfy certain axioms known as Ruan's axioms (omitted here).
- **Intuition:** Since $\mathbb{M}_n(X) \cong \mathbb{M}_n \otimes X$, one can think of an operator space as an assignment of a norm on each space $\mathbb{M}_n \otimes X$.
- **Examples:** all von Neumann algebras and their (pre)duals are operator spaces. In particular, the spaces M_n, T_n, all objects relevant to both the Heisenberg and Schrödinger picture.
- A linear map $\varphi : X \to Y$ between operator spaces is called a *complete contraction* if each function $\varphi_n : M_n(X) \to M_n(Y)$ is a contraction.
- **Intuition:** One may think of this as saying that $\mathrm{id}_{M_n} \otimes f : \mathbb{M}_n \otimes X \to \mathbb{M}_n \otimes Y$ is a contraction. Obvious similarity with completely positive maps.

Higher-order Quantum Computation, Operator Spaces, and more

- Let **FdOS** be the category of operator spaces with complete contractions as morphisms.
- It is symmetric monoidal closed and it has finite (co)products. Therefore a model of a linear lambda calculus.
- Moreover, it is also $*$-autonomous and therefore a model of MALL (multiplicative additive linear logic).
- **Problem:** The morphisms do not have to be completely positive.
- **Solution 1:** Use a categorical construction, called *double gluing* on **FdOS**. This gives a model **Q** of MALL for quantum computation which respects the Heisenberg-Schrödinger duality. Recent paper in FoSSaCS 2026 with Thea Li.
- **Solution 2:** No double gluing, but even more mathematics... PhD project of Sandro Rodríguez Muñoz.

Heisenberg Picture, von Neumann Algebras, and Negative Logical Polarity

Heisenberg Picture	$\mathbf{vNAlg} \overset{\text{full}}{\hookrightarrow} \mathbf{Q}$	LL_{-}
System description	$\mathcal{A}, \mathcal{B}$	N, M
Quantum composition	$\mathcal{A} \mathbin{\tilde{\otimes}} \mathcal{B}$	$N \parr M$
Classical composition	$\mathcal{A} \overset{\infty}{\oplus} \mathcal{B}$	$N \mathbin{\&} M$
Quantum operation	$\mathcal{B} \xrightarrow{\text{CPU}} \mathcal{A}$	$M \vdash N$

Schrödinger Picture, von Neumann Coalgebras, and Positive Logical Polarity

Schrödinger Picture	$\mathbf{vNCoalg} \overset{\text{full}}{\hookrightarrow} \mathbf{Q}$	LL_+
System description	$\mathcal{C}, \mathcal{D}$	P, R
Quantum composition	$\mathcal{C} \mathbin{\widehat{\otimes}} \mathcal{D}$	$P \otimes R$
Classical composition	$\mathcal{C} \overset{1}{\oplus} \mathcal{D}$	$P \oplus R$
Quantum operation	$\mathcal{C} \xrightarrow{\text{CPTP}} \mathcal{D}$	$P \vdash R$

Conclusion

- Inspired by Curry-Howard-Lambek, we search for good mathematical models of quantum computing.
- Clean models of higher-order quantum computation require more refined mathematics than basic linear algebra, e.g. categorical constructions (double gluing) or functional analysis (operator spaces, von Neumann algebras) or something else?
- Designing the higher-order structure around the Heisenberg-Schrödinger duality is very natural. Recall that $A \multimap B \equiv A^{\perp} \parr B \equiv (A \otimes B^{\perp})^{\perp}$.
- **Future work:** A model of full linear logic built around the Heisenberg-Schrödinger duality for infinite-dimensional quantum computing. No way to escape functional analysis in my view.
- **Future work:** Design higher-order languages where we move away from qubit-level programming.

References and Further Reading

Most of the presented material is already known.

- Operator algebras and operator spaces:
  - Masamichi Takesaki. **Theory of Operator Algebras I.**
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- Category theory and semantics:
  - Chris Heunen, Jamie Vicary. **Categories for Quantum Theory: An Introduction.**
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- Relevant papers:
  - Bert Lindenhovius, Vladimir Zamdzhiev. **The Category of Operator Spaces and Complete Contractions.** 2024.
  - Bert Lindenhovius, Vladimir Zamdzhiev. **Operator Spaces, Linear Logic and the Heisenberg-Schrödinger Duality of Quantum Theory.** LICS 2025.
  - Thea Li, Vladimir Zamdzhiev. **Quantum Coherence Spaces Revisited: A von Neumann (Co)Algebraic Approach.** FoSSaCS 2026.